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THE THEORY OF PIEZOELECTRIC SHELLS*

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A variational formulation of the problem of the equilibrium of completely anisotropic piezoelectric shells is formulated. Using a variational asymptotic method, a system of two-dimensional statics equations is derived for piezoelectric shells. The asymptotic accuracy of the two-dimensional theory constructed is proved.

1. Variational principles of the theory of piezoelectricity. In a three-dimensional space R_3 we consider a linear dielectric body occupying the domain V in the initial state. We introduce in V Lagrange coordinates of the points ξ^a which are related to the Cartesian coordinates x^i by the equations $x^i = x^i(\xi^a)$. Let the body boundary ∂V be the union of two-dimensional surfaces $S_\varphi^{(1)}, \ldots, S_\varphi^{(N)}$ (electrodes) and the rest of the boundary S_τ. For simplicity we will consider the case of purely electrical loading on the body, which corresponds to giving the value of the electric potential on the electrodes. Then the fundamental variational principle of the electrostatics of dielectrics states /1/: among all possible displacement fields w_i and all possible electric induction fields D^a that satisfy the conditions

$$\nabla_a D^a = 0 \quad \text{in} \quad V, \quad D^a \nu_a = 0 \quad \text{on} \quad S_\tau \tag{1.1}$$

the true functions $\bar{w}_i, \bar{D}^a$ at the equilibrium position yield a minimum of the body energy functional

$$I = \int_V U(\varepsilon_{ab}, D_a)\, dv + \sum_{n=1}^{N} \varphi_n \int_{S_\varphi^{(n)}} D_a \nu^a\, d\omega \tag{1.2}$$

The ν_a in (1.1), (1.2) are components of the external normal vector to ∂V, dv is an element of volume, $d\omega$ is an element of area, $\varphi_n = \mathrm{const}$ is the value of the potential on the n-th electrode $S_\varphi^{(n)}$, and $\varepsilon_{ab} = x_{(a}{}^i w_{i,b)}$ are strain tensor components $(x_a{}^i = \partial x^i/\partial \xi^a)$. Here an henceforth the superscripts $a, b, c, d, \ldots$ correspond to projections on the axes of the accompanying coordinate system ξ^a, while the superscripts $i, j, k, l, \ldots$ are projections on the axes of the Cartesian coordinate system x^i. The comma in the subscripts denotes partial differentiation, the symbol ∇_a represents covariant differentiation in the metric g_{ab}, and the parentheses in the subscripts denote the symmetrization operation. The subscripts and superscripts are juggled by using the metric g_{ab}, while summation is over repeated upper and lower indices.

A piezoelectric is a dielectric body for which the function $U(\varepsilon_{ab}, D_a)$ is a strictly positive quadratic form in ε_{ab} and D_a

$$U(\varepsilon_{ab}, D_a) = \frac{1}{2} c_D^{abcd} \varepsilon_{ab}\varepsilon_{cd} - h^{abc}\varepsilon_{bc} D_a + \frac{1}{2} \beta_S^{ab} D_a D_b \tag{1.3}$$

By varying the functional (1.2) under the constraints (1.1) for this case, we obtain the equations and boundary conditions

$$\nabla_a D^a = 0, \quad E_a = -\nabla_a \varphi, \quad \nabla_b \sigma^{ab} = 0, \quad \varepsilon_{ab} = x_{(a}^i w_{i,b)} \tag{1.4}$$
$$\sigma^{ab} = \frac{\partial U}{\partial \varepsilon_{ab}} = c_D^{abcd}\varepsilon_{cd} - h^{cab} D_c, \quad E^a = \frac{\partial U}{\partial D_a} = -h^{abc}\varepsilon_{bc} + \beta_S^{ab} D_b$$

$$D^a \nu_a = 0 \quad \text{on} \quad S_\tau, \quad \varphi = \varphi_n \quad \text{on} \quad S_\varphi^{(n)}, \quad n = 1, \ldots, N \tag{1.5}$$
$$\sigma^{ab}\nu_b = 0 \quad \text{on} \quad \partial V$$

Here E_a is the electric field, φ is the electric potential (essentially the Lagrange multiplier for the constraint (1.1)), and σ^{ab} are stress tensor components.

Applying the reciprocity technique /2/, the principles (1.1) and (1.2) can be reformulated into the following minimax principle: the true functions $\bar{w}_i, \bar{\varphi}$ at the equilibrium position yield a maximum in φ and a minimum in w_i for the functional

$$I = \int_V H(\varepsilon_{ab}, E_a)\, dv \tag{1.6}$$

under the constraints

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$$\varphi = \varphi_n \quad \text{on} \quad S_\varphi^{(n)},\ n = 1, \ldots, N \tag{1.7}$$

where $E_a = -\nabla_a \varphi$. The function $H(\varepsilon_{ab}, E_a)$, called the electric enthalpy density /3/, is the Legendre transformation of the function $U(\varepsilon_{ab}, D_a)$ in the variable D_a taken with opposite sign. If U is given by (1.3), then

$$H = \frac{1}{2} c_E^{abcd} \varepsilon_{ab} \varepsilon_{cd} - e^{abc} \varepsilon_{bc} E_a - \frac{1}{2} \epsilon_S^{ab} E_a E_b \tag{1.8}$$

$$(\epsilon_S^{ab} = (\beta_S^{ab})^{-1},\ e^{abc} = \epsilon_S^{ad} h_d^{\cdot bc},\ c_E^{abcd} = c_D^{abcd} - e_g^{\cdot ab} h^{gcd})$$

According to the definition of $H(\varepsilon_{ab}, E_a)$ we have the relationship

$$\sigma^{ab} = \frac{\partial H}{\partial \varepsilon_{ab}} = c_E^{abcd} \varepsilon_{cd} - e^{cab} E_c, \quad D^a = -\frac{\partial H}{\partial E_a} = e^{abc} \varepsilon_{bc} + \epsilon_S^{ab} E_b \tag{1.9}$$

which together with the statics Eqs.(1.4) and the boundary conditions (1.5) form a correctly formulated boundary value problem.

We shall later consider the minimax principle (1.6), (1.7) most convenient for the application of a variational asymptotic analysis in the equilibrium problem for piezoelectric shells. See /4, 5/ for other variational principles in the theory of piezoelectricity.

2. Three-dimensional problems of the theory of piezoelectric shells. Let us consider a domain V of the special form

$$x^i(\xi^\alpha, \xi^3) = r^i(\xi^\alpha) + \xi^3 n^i(\xi^\alpha) \tag{2.1}$$

in R_3, where $x^i = r^i(\xi^\alpha)$ is the equation of a smooth surface Ω bounded by the contour Γ, and n^i are components of the unit vector normal to Ω. The coordinates ξ^α, ξ^3 vary in a cylinder of height h: $\xi^\alpha \in \Omega$, $|\xi^3| \leqslant h/2$; the domain of variation of ξ^α, exactly like the middle surface, is denoted by Ω; the small Greek superscripts run through the values 1, 2 and correspond to projections on the ξ^α axis while the superscript 3 is usually omitted ($\xi^3 \equiv \xi$). A piezoelectric body occupying the domain V in the initial state is called a piezoelectric shell with middle surface Ω and thickness h.

Let $\Omega_\pm$ denote the facial surfaces of the shell given by (2.1) for $\xi = \pm h/2$. We will examine the two methods of electrical loading of a shell encountered most often /3/.

A. There are no electrodes on the facial surfaces. There are electrodes on the shell edge, i.e., the contour Γ is a union of curves $\Gamma_\varphi^{(1)}, \ldots, \Gamma_\varphi^N$ (where there are electrodes) and the rest Γ_τ. For $\xi^\alpha \in \Gamma_\varphi^{(n)} \times [-h/2, h/2]$ values are given for the electric potential

$$\varphi = \varphi_n,\ n = 1, 2, \ldots, N \tag{2.2}$$

B. The facial surfaces $\Omega_\pm$ are covered with electrodes. Values of the electric potential are given on them

$$\varphi = \pm\varphi_0/2 \quad \text{for} \quad \xi = \pm h/2 \tag{2.3}$$

According to the variational principle (1.6), (1.7), the true displacements $\bar{w}_i$ and the electric potential $\bar{\varphi}$ correspond to extremals of the functional

$$I = \int_\Omega \int_{-h/2}^{h/2} H(\varepsilon_{ab}, E_a) \varkappa \, d\xi \, d\omega \tag{2.4}$$

under the constraints (2.2) (in problem A), and the constraints (2.3) (in problem B).

The $H(\varepsilon_{ab}, E_a)$ in the functional (2.4) is the electric enthalpy density given by (1.8) $\varkappa = 1 - 2H\xi + K\xi^2$, $d\omega = \sqrt{a}\, d\xi^1 d\xi^2$, $a = \det \| a_{\alpha\beta} \|$, where $a_{\alpha\beta}$ is the first quadratic form (metric) of Ω, H and K are the mean and Gaussian curvatures of Ω. The following geometric and kinematic relationships /2/ are true in the coordinate system (2.1):

$$g_{\alpha\beta} = a_{\alpha\beta} - 2b_{\alpha\beta}\xi + c_{\alpha\beta}\xi^2, \quad g_{\alpha3} = 0, \quad g_{33} = 1 \tag{2.5}$$

$$g^{\alpha\beta} = \frac{1}{\varkappa^2}[(1 - 2H\xi) a^{\alpha\beta} + 2\xi(1 - 2H\xi) b^{\alpha\beta} + \xi^2 c^{\alpha\beta}],$$

$$g^{\alpha3} = 0, \quad g^{33} = 1$$

$$\varepsilon_{\alpha\beta} = x^i_{(\alpha} w_{i,\beta)} = r^i_{(\alpha} w_{i,\beta)} - \xi b^\lambda_{(\alpha} r_\lambda^{\ i} w_{i,\beta)}, \quad \varepsilon_{33} = n^i w_{i,\xi}$$

$$2\varepsilon_{\alpha3} = x_\alpha^{\ i} w_{i,\xi} + n^i w_{i,\alpha} = r_\alpha^{\ i} w_{i,\xi} - \xi b_\alpha^{\ \lambda} r_\lambda^{\ i} w_{i,\xi} + n^i w_{i,\alpha}$$

where $b_{\alpha\beta}$, $c_{\alpha\beta}$ are the second and third quadratic forms of Ω.

The problem is to replace the three-dimensional functional (2.4) by an approximate two-dimensional functional in which there are functions dependent only on the longitudinal coordinates ξ^1, ξ^2.

The possibility of changing from the three- to the two-dimensional problem is related to the smallness of the ratio between the thickness h and the characteristic radius of curvature R of the shell middle surface /2/, and to the characteristic scale and strain and electrical field variation over the longitudinal coordinates l. By using a variational asymptotic method

/2/, a two-dimensional functional will be constructed below for the electrical enthalpy of piezoelectric shells in which terms of the order of h/R and h/l are neglected compared with unity (the "classical" approximation). Extending the technique of estimating the error /6–8/ to the statics of piezoelectrics, we prove a theorem according to which the theory of piezoelectric shells constructed in this paper actually allows an error of $h/R + h/l$ in determining the electroelastic state of stress.

Two-dimensional theories have been constructed in /9–15/ in certain special cases of piezoelectric shells (see the survey /16/, also). The papers /9, 10, 14, 15/ are devoted to piezoelectric shells, and /11–13/ to piezoelectric plates. An asymptotically exact theory of completely anisotropic piezoelectric shells was constructed in /17/ (the case of piezoelectric shells with electrodes on the facial surfaces is examined in addition in this paper).

3. Two-dimensional moduli. We represent the electrical enthalpy density $H(\varepsilon_{ab}, E_a)$ in the following form:

$$H = H_{\parallel} + H_{\perp}; \quad H_{\parallel} = \min_{\varepsilon_{\alpha 3},\, \varepsilon_{33}} \max_{E_3} H \tag{3.1}$$

The representation of $H(\varepsilon_{ab}, E_a)$ in the form (3.1) turns out to be convenient for asymptotic analysis of the functional (2.4). Simple calculations show that

$$H_{\parallel} = \frac{1}{2} c_N^{\alpha\beta\gamma\delta} \varepsilon_{\alpha\beta}\varepsilon_{\gamma\delta} - e_N^{\gamma\alpha\beta}\varepsilon_{\alpha\beta}E_\gamma - \frac{1}{2}\epsilon_N^{\alpha\beta}E_\alpha E_\beta \tag{3.2}$$

$$\begin{aligned} H_{\perp} &= \frac{1}{2} c_E^{3333}\gamma^2 + c_E^{\alpha 333}\gamma\gamma_\alpha + \frac{1}{2} c_E^{3\alpha 3\beta}\gamma_\alpha\gamma_\beta - e^{333}\gamma F - \\ &\quad e^{3\alpha 3}\gamma_\alpha F - \frac{1}{2}\epsilon_S{}^{33}F^2 \\ \gamma &= \varepsilon_{33} + r^{\alpha\beta}\varepsilon_{\alpha\beta} - r^\alpha E_\alpha, \quad \gamma_\alpha = 2\varepsilon_{\alpha 3} + t_\alpha^{\mu\nu}\varepsilon_{\mu\nu} - t_\alpha{}^\mu E_\mu \\ F &= E_3 + q^{\alpha\beta}\varepsilon_{\alpha\beta} + q^\alpha E_\alpha \end{aligned} \tag{3.3}$$

The coefficients $c_N^{\alpha\beta\gamma\delta}$, $e_N^{\gamma\alpha\beta}$, $\epsilon_N{}^{\alpha\beta}$, $c_E^{\alpha 333}$, c_E^{3333}, e^{333} $e^{3\alpha 3}$, $\epsilon_S{}^{33}$, $r^{\alpha\beta}$, r^α, $t_\alpha{}^{\mu\nu}$, $t_\alpha{}^\mu$, $q^{\alpha\beta}$, q^α in the transformations of the coordinate systems on the middle surface behave as surface tensors. We shall call them "two-dimensional" moduli. They are evaluated in terms of the three-dimensional moduli by means of the formulas

$$\begin{aligned} & c_N^{\alpha\beta\gamma\delta} = c_P^{\alpha\beta\gamma\delta} + q^{\alpha\beta}e_P^{3\gamma\delta}, \quad e_N^{\gamma\alpha\beta} = e_P^{\gamma\alpha\beta} - q^{\alpha\beta}\epsilon_P^{\gamma 3} \\ & \epsilon_N^{\alpha\beta} = \epsilon_P^{\alpha\beta} - q^\alpha\epsilon_P^{\beta 3}, \quad q^{\alpha\beta} = e_P^{3\alpha\beta}/\epsilon_P^{33}, \quad q^\alpha = \epsilon_P^{\alpha 3}/\epsilon_P^{33} \\ & c_P^{\alpha\beta\gamma\delta} = c^{\alpha\beta\gamma\delta} - k_\nu^{\alpha\beta}\bar{c}^{\gamma\delta\nu 3}, \quad e_P^{a\alpha\beta} = \bar{e}^{a\alpha\beta} - k_\nu^{\alpha\beta}\bar{e}^{a\nu 3} \\ & \epsilon_P^{ab} = \bar{\epsilon}^{ab} + k_\nu{}^a\bar{e}^{b\nu 3}, \quad \epsilon_P^{33} = \bar{\epsilon}^{33} + k_\nu\bar{e}^{3\nu 3} \\ & k_\alpha^{\mu\nu} = H_{\alpha\beta}\bar{c}^{\mu\nu\beta 3}, \quad k_\alpha{}^\mu = H_{\alpha\beta}\bar{e}^{\mu\beta 3}, \quad k_\alpha = H_{\alpha\beta}\bar{e}^{3\beta 3}, \quad H_{\alpha\beta} = (\bar{c}^{3\alpha 3\beta})^{-1} \\ & \bar{c}^{a\alpha b\beta} = c_E^{a\alpha b\beta} - c_E^{a\alpha 33}c_E^{b\beta 33}/c_E^{3333}, \quad \bar{e}^{ab\beta} = e^{ab\beta} - c_E^{b\beta 33}e^{a33}/c_E^{3333} \\ & \bar{\epsilon}^{ab} = \epsilon_S^{ab} + e^{a33}e^{b33}/c_E^{3333}; \quad t_\alpha^{\mu\nu} = k_\alpha^{\mu\nu} + k_\alpha q^{\mu\nu} \\ & t_\alpha{}^\mu = k_\alpha{}^\mu - k_\alpha q^\mu, \quad r^{\alpha\beta} = f^{\alpha\beta} + fq^{\alpha\beta}, \quad r^\alpha = f^\alpha + fq^\alpha \\ & f^{\alpha\beta} = \frac{c_E^{\alpha\beta 33} - c_E^{\lambda 333}k_\lambda^{\alpha\beta}}{c_E^{3333}}, \quad f^\alpha = \frac{e^{\alpha 33} - c_E^{\lambda 333}k_\lambda{}^\alpha}{c_E^{3333}}, \quad f = \frac{e^{333} - c_E^{\lambda 333}k_\lambda}{c_E^{3333}} \end{aligned} \tag{3.4}$$

For simplicity we shall consider the case of piezoelectric shells that are homogeneous over the thickness. It can be shown that for such shells any two-dimensional moduli possess the properties

$$\mathbf{A}(\xi^\alpha, \xi) = \overline{\mathbf{A}}(\xi^\alpha) + O\left(\frac{h}{R}\right)\overline{\mathbf{A}}(\xi^\alpha)$$

Therefore, when constructing a classical theory of shells having the errors h/R and h/l as compared with unity, it can be assumed that $\mathbf{A} = \overline{\mathbf{A}}$, i.e., the two-dimensional moduli of shells homogeneous over the thickness are independent of the transverse coordinate.

We will distinguish certain special symmetry cases.

1°. *Symmetry planes parallel to the middle surface.* If the properties of the medium are invariant under reflections relative to planes parallel to the middle surface, then the following two-dimensional tensors vanish:

$$c_E^{\alpha 333} = 0, \quad e^{333} = 0, \quad t_\alpha^{\mu\nu} = 0, \quad t_\alpha{}^\mu = 0, \quad q^{\mu\nu} = 0, \quad q^\mu = 0$$

2°. *Transversal isotropy.* When the properties of the medium are invariant under rotation around the vector normal to the middle surface (a model of a piezoceramic shell polarized along the normal with symmetry $\infty\cdot m$ /3/), it can be shown that all two-dimensional tensors with odd number of superscripts vanish, the tensor $c_N^{\alpha\beta\gamma\delta}$ has the form

$$c_N^{\alpha\beta\gamma\delta} = c_1{}^N a^{\alpha\beta}a^{\gamma\delta} + c_2{}^N(a^{\alpha\gamma}a^{\beta\delta} + a^{\alpha\delta}a^{\beta\gamma})$$

and all the two-dimensional tensors of second rank are global.

4. Asymptotic analysis of the electrical enthalpy functional. *Problem A.* In order to fix the domain of variation of the transverse coordinate in the passage to the limit $h \to 0$, we make the change of variable $\xi = h\zeta$, $|\zeta| \leqslant 1/_2$. Then h will occur explicitly in the functional (2.4). We shall seek the field of the displacement w^i and the potential φ in the form

$$w^i = u^i - hr_\alpha{}^i\psi^\alpha\zeta + hy^i(\xi^\alpha, \zeta), \quad \psi_\alpha = n^i u_{i,\alpha} \qquad (4.1)$$
$$\varphi = \psi + h\chi(\xi^\alpha, \zeta)$$

where u^i, ψ are independent of ζ. Because of the overdefinition of u^i, ψ the following constraints can be imposed on y^i, χ :

$$\langle y^i \rangle = 0, \ \langle \chi \rangle = 0 \qquad (4.2)$$

where $\langle\cdot\rangle$ is the integral with respect to ζ within the limits $[-1/_2, 1/_2]$. Eqs.(4.1) and (4.2) set up a mutually one-to-one correspondence between w^i, φ and the set of functions u^i, ψ, y^i, χ and determine the change in the desired functions $\{w^i, \varphi\} \to \{u^i, \psi, y^i, \chi\}$.

Asymptotic analysis enables us to determine the order of smallness of y^i, χ. If these terms are neglected, then (4.1) is a generalization of the well-known Kirchhoff-Love hypotheses to a piezoelectric shell. The electroelastic state of stress of a shell is here characterized completely by the measure of the tension $A_{\alpha\beta} = r_{(\alpha}{}^i u_{i,\beta)}$, by the measure of the bending $B_{\alpha\beta} = \psi_{(\alpha;\beta)} + b_{(\alpha}{}^\lambda r_\lambda{}^i u_{i,\beta)}$ and by the surface electric field $F_\alpha = -\psi_{,\alpha}$; the covariant differentiation in the metric $a_{\alpha\beta}$ is denoted by the semicolon in the subscripts.

We introduce the following notation:

$$e_A = \max_\Omega (A_{\alpha\beta}A^{\alpha\beta})^{1/2}, \quad e_B = \frac{h}{2}\max_\Omega (B_{\alpha\beta}B^{\alpha\beta})^{1/2},$$
$$f_F = \max_\Omega (F_\alpha F^\alpha)^{1/2}$$
$$y_\alpha = r_\alpha{}^i y_i, \quad y = n^i y_i$$
$$\Delta_\alpha = \max_V |y_{\alpha,\zeta}|, \quad \Delta = \max_V |y_{,\zeta}|, \quad \Pi = \max_V |\chi_{,\zeta}|$$

We consider a certain point of Ω. The best constant l in the inequalities

$$|A_{\alpha\beta,\gamma}| \leqslant \frac{e_A}{l}, \quad h|B_{\alpha\beta,\gamma}| \leqslant \frac{e_B}{l}, \quad |F_{\alpha,\beta}| \leqslant \frac{f_F}{l} \qquad (4.3)$$
$$\max_\zeta |y_{\alpha,\beta}| \leqslant \frac{\Delta_\alpha}{l}, \quad \max_\zeta |y_{,\alpha}| \leqslant \frac{\Delta}{l}, \quad \max_\zeta |\chi_{,\alpha}| \leqslant \frac{\Pi}{l}$$

is called the characteristic scale of variation of the electric field deformation in the longitudinal coordinates. We define the internal domain Ω_2 as a subdomain of Ω in which the following inequalities hold:

$$h_* = h/R \ll 1, \ h_{**} = h/l \ll 1 \qquad (4.4)$$

We assume the domain Ω to consist of the interior domain Ω_2 and the domain Ω_1 abutting on the contour Γ with width of order h (boundary layer). Then the functional (2.4) can be divided into the sum of two functionals, an inner one for which an iteration process will be constructed, and a boundary layer. As in the theory of elastic shells /2/, the boundary layer functional can be neglected in the classical approximation. Therefore, the problem reduces to finding the minimax point of the interior functional that can be identified with the functional (2.4) $(\Omega_2 \equiv \Omega)$.

We fix u^i, ψ and we seek y_i, χ. It follows from the assumption (4.4) and formulas (2.6) and (4.3) that to a first approximation

$$\varepsilon_{\alpha\beta} = A_{\alpha\beta} - hB_{\alpha\beta}\zeta, \quad 2\varepsilon_{\alpha 3} = y_{\alpha,\zeta}, \quad \varepsilon_{33} = y_{,\zeta}, \quad E_\alpha = F_\alpha, \qquad (4.5)$$
$$E_3 = -\chi_{,\zeta}$$

Substituting (4.5) into the functional (2.4) with an electric enthalpy density in the form (3.1), (3.2), we obtain the functional

$$I = \int_\Omega [\Psi(A_{\alpha\beta}, B_{\alpha\beta}, F_\alpha) + J_\perp(y_\alpha, y, \chi)]\, d\omega \qquad (4.6)$$
$$\Psi = \frac{h}{2}\langle c_N^{\alpha\beta\gamma\delta}(A_{\alpha\beta} - hB_{\alpha\beta}\zeta)(A_{\gamma\delta} - hB_{\gamma\delta}\zeta) -$$
$$2e_N^{\gamma\alpha\beta}(A_{\alpha\beta} - hB_{\alpha\beta}\zeta)F_\gamma - \epsilon_N^{\alpha\beta}F_\alpha F_\beta\rangle$$
$$J_\perp = \frac{h}{2}\langle c_E^{3333}\gamma^2 + 2c_E^{\alpha 333}\gamma\gamma_\alpha + c_E^{3\alpha 3\beta}\gamma_\alpha\gamma_\beta - 2e^{333}\gamma F - 2e^{3\alpha 3}\gamma_\alpha F - \epsilon_S^{33}F^2\rangle$$
$$\gamma = y_{,\zeta} + r^{\alpha\beta}(A_{\alpha\beta} - hB_{\alpha\beta}\zeta) - r^\alpha F_\alpha$$

$$\gamma_\alpha = y_{\alpha,\zeta} + t_\alpha^{\mu\nu}(A_{\mu\nu} - hB_{\mu\nu}\zeta) - t_\alpha{}^\mu F_\mu$$
$$F = -\chi_{,\zeta} + q^{\alpha\beta}(A_{\alpha\beta} - hB_{\alpha\beta}\zeta) + q^\alpha F_\alpha$$

The functions y_α, y, χ only occur in the functional $J_\perp$ in terms of γ, γ_α, F.

We minimize the functional (4.6) in y_α, y and maximize in χ under the constraints (4.2). The minimax value I is obviously reached for $\gamma = \gamma_\alpha = F \equiv 0$, i.e., for

$$y = -(r^{\alpha\beta}A_{\alpha\beta} - r^\alpha F_\alpha)\zeta + \frac{1}{2}hr^{\alpha\beta}B_{\alpha\beta}\left(\zeta^2 - \frac{1}{12}\right) \tag{4.7}$$
$$y_\alpha = -(t_\alpha^{\mu\nu}A_{\mu\nu} - t_\alpha{}^\mu F_\mu)\zeta + \frac{1}{2}ht_\alpha^{\mu\nu}B_{\mu\nu}\left(\zeta^2 - \frac{1}{12}\right)$$
$$\chi = (q^{\alpha\beta}A_{\alpha\beta} + q^\alpha F_\alpha)\zeta - \frac{1}{2}hq^{\alpha\beta}B_{\alpha\beta}\left(\zeta^2 - \frac{1}{12}\right)$$

The functional $J_\perp$ vanishes on the extremals and the average electrical enthalpy functional takes the form

$$J = \int_\Omega \Psi(A_{\alpha\beta}, B_{\alpha\beta}, F_\alpha)\,d\omega \tag{4.8}$$
$$\Psi(A_{\alpha\beta}, B_{\alpha\beta}, F_\alpha) = \frac{h}{2}\Big(c_N^{\alpha\beta\gamma\delta}A_{\alpha\beta}A_{\gamma\delta} + \frac{h^2}{12}c_N^{\alpha\beta\gamma\delta}B_{\alpha\beta}B_{\gamma\delta} -$$
$$2e_N^{\gamma\alpha\beta}A_{\alpha\beta}F_\gamma - \epsilon_N^{\alpha\beta}F_\alpha F_\beta\Big)$$

Problem B. In this problem the electric potential φ should satisfy the constraints (2.3), consequently, we make a different change in the desired functions than (4.1) by substituting $\varphi_0\zeta$ in place of ψ. We impose the constraint

$$\langle y^i\rangle = 0, \quad \chi|_{\zeta=\pm 1/2} = 0 \tag{4.9}$$

on the functions y^i, χ

By performing a procedure analogous to asymptotic analysis, it can be shown that the relationships

$$\varepsilon_{\alpha\beta} = A_{\alpha\beta} - hB_{\alpha\beta}\zeta, \quad 2\varepsilon_{\alpha 3} = y_{\alpha,\zeta}, \quad \varepsilon_{33} = y_{,\zeta} \tag{4.10}$$
$$E_\alpha = 0, \; E_3 = -\varphi_0/h - \chi_{,\zeta}$$

are true to a first approximation.

Substituting (4.10) into the functional (2.4) and seeking its minimal value in y_α, y, χ, we obtain the two-dimensional functional

$$J = \int_\Omega [\Psi_\parallel(A_{\alpha\beta}, B_{\alpha\beta}) + \Psi_\perp(A_{\alpha\beta}, B_{\alpha\beta})]\,d\omega \tag{4.11}$$

$$\Psi_\parallel = \frac{h}{2}\Big(c_N^{\alpha\beta\gamma\delta}A_{\alpha\beta}A_{\gamma\delta} + \frac{h^2}{12}c_N^{\alpha\beta\gamma\delta}B_{\alpha\beta}B_{\gamma\delta}\Big) \tag{4.12}$$

$$\Psi_\perp = \inf_{y_\alpha, y}\sup_\chi J_\perp = \inf_{y_\alpha, y}\sup_\chi \frac{h}{2}\langle c_E^{3333}\gamma^2 + 2c_E^{\alpha 333}\gamma\gamma_\alpha +$$
$$c_E^{3\alpha 3\beta}\gamma_\alpha\gamma_\beta - 2e^{333}\gamma F - 2e^{3\alpha 3}\gamma_\alpha F - \epsilon_S^{33}F^2\rangle$$
$$\gamma = y_{,\zeta} + r^{\alpha\beta}(A_{\alpha\beta} - hB_{\alpha\beta}\zeta), \quad \gamma_\alpha = y_{\alpha,\zeta} + t_\alpha^{\mu\nu}(A_{\mu\nu} - hB_{\mu\nu}\zeta)$$
$$F = -\varphi_0/h - \chi_{,\zeta} + q^{\alpha\beta}(A_{\alpha\beta} - hB_{\alpha\beta}\zeta)$$

It can be shown that the minimax value $J_\perp$ in (4.12) under the constraints (4.9) will be reached if and only if

$$F = -\frac{\varphi_0}{h} + q^{\alpha\beta}A_{\alpha\beta}, \quad \gamma_\alpha = k_\alpha F, \quad \gamma = fF \tag{4.13}$$
$$\chi = -\frac{1}{2}hq^{\alpha\beta}B_{\alpha\beta}\left(\zeta^2 - \frac{1}{4}\right)$$
$$y = -f\frac{\varphi_0}{h}\zeta - f^{\alpha\beta}A_{\alpha\beta}\zeta + \frac{1}{2}hr^{\alpha\beta}B_{\alpha\beta}\left(\zeta^2 - \frac{1}{12}\right)$$
$$y_\alpha = -k_\alpha\frac{\varphi_0}{h}\zeta - k_\alpha^{\mu\nu}A_{\mu\nu}\zeta + \frac{1}{2}ht_\alpha^{\mu\nu}B_{\mu\nu}\left(\zeta^2 - \frac{1}{12}\right)$$

On the extremals (4.13)

$$\Psi_\perp = -\frac{h}{2}\epsilon_P^{33}F^2 = -\frac{h}{2}\epsilon_P^{33}\left(-\frac{\varphi_0}{h} + q^{\alpha\beta}A_{\alpha\beta}\right)^2$$

Therefore, taking (3.4) into account, we can write the average electrical enthalpy functional for shells with electrodes on the facial surfaces in the final form

$$J = \int_\Omega \Psi(A_{\alpha\beta}, B_{\alpha\beta})\,d\omega \tag{4.14}$$

$$\Psi(A_{\alpha\beta}, B_{\alpha\beta}) = \frac{h}{2}\Big(c_P^{\alpha\beta\gamma\delta}A_{\alpha\beta}A_{\gamma\delta} + \frac{h^2}{12}c_N^{\alpha\beta\gamma\delta}B_{\alpha\beta}B_{\gamma\delta} + 2e_P^{3\alpha\beta}A_{\alpha\beta}\frac{\varphi_0}{h}\Big)$$

Within the framework of classical approximation accuracy, a certain simplification in (4.8) and (4.14) can be achieved by, for instance, replacing the bending measure $B_{\alpha\beta}$ by another measure $\tilde{B}_{\alpha\beta} = B_{\alpha\beta} + Q_{\alpha\beta}^{\gamma\delta}A_{\gamma\delta}$, where the tensor $Q_{\alpha\beta}^{\gamma\delta}$ depends only on $a_{\alpha\beta}$ and $b_{\alpha\beta}$. In particular, the following measures were proposed /2/: Koiter-Sanders $\rho_{\alpha\beta}=B_{\alpha\beta}-b_{(\alpha}^{\lambda}A_{\lambda\beta)}$ and Novozhilov-Balabukh $\varkappa_{\alpha\beta}=-\rho_{\alpha\beta}-{}^1\!/_2\bar{H}d_{\alpha\beta}d^{\gamma\delta}A_{\gamma\delta}$ where $d_{\alpha\beta}=\tilde{b}_{\alpha\beta}/\sqrt{\tilde{b}}$, $\tilde{b}={}^1\!/_2\tilde{b}_{\alpha\beta}\tilde{b}^{\alpha\beta}$, $\tilde{b}_{\alpha\beta}=b_{(\alpha}^{\gamma}e_{\gamma\beta)}$. In this connection diverse invariants of the classical theory of piezoelectric shells can be obtained /9, 10, 14, 15—17/.

5. Equilibrium equation of piezoelectric shells. *Case* A. The electrical enthalpy functional is given by (4.8). It follows from condition (2.2) that to a first approximation the function ψ should satisfy the constraints

$$\psi=\varphi_n \quad \text{on} \quad \Gamma_\varphi^{(n)},\ n=1,\ldots,N \tag{5.1}$$

By varying the functional (4.8) under the constraints (5.1), we obtain the system of equations

$$\begin{gathered}T^{\alpha\beta}_{;\beta}-b_\lambda{}^\alpha M^{\lambda\beta}_{;\beta}=0,\quad M^{\alpha\beta}_{;\alpha\beta}+T^{\alpha\beta}b_{\alpha\beta}=0\\ G^{\alpha}_{;\alpha}=0,\quad T^{\alpha\beta}=S^{\alpha\beta}-b_\sigma{}^\alpha M^{\sigma\beta}\end{gathered} \tag{5.2}$$

The $S^{\alpha\beta}$ in (5.2) is the tensile force tensor, $M^{\alpha\beta}$ is the bending moment tensor, and G^α is the "surface" electric induction vector. They are expressed in terms of $A_{\alpha\beta}$, $B_{\alpha\beta}$, F_α by the following electroelasticity relationships:

$$\begin{gathered}S^{\alpha\beta}=\frac{\partial\Psi}{\partial A_{\alpha\beta}}=h\left(c_N^{\alpha\beta\gamma\delta}A_{\gamma\delta}-e_N^{\gamma\alpha\beta}F_\gamma\right),\\ M^{\alpha\beta}=-\frac{\partial\Psi}{\partial B_{\alpha\beta}}=-\frac{h^3}{12}c_N^{\alpha\beta\gamma\delta}B_{\gamma\delta},\\ G^\alpha=-\frac{\partial\Psi}{\partial F_\alpha}=h\left(e_N^{\alpha\beta\gamma}A_{\beta\gamma}+\epsilon_N^{\alpha\beta}F_\beta\right),\end{gathered} \tag{5.3}$$

Together with the kinematic relationships

$$\begin{gathered}A_{\alpha\beta}=u_{(\alpha;\beta)}-b_{\alpha\beta}u,\quad B_{\alpha\beta}=u_{;\alpha\beta}-c_{\alpha\beta}u+2b^{\lambda}_{(\alpha}u_{\lambda;\beta)}+b^{\lambda}_{\alpha;\lambda}u_\lambda\\ F_\alpha=-\psi_{,\alpha}\end{gathered} \tag{5.4}$$

(5.2) and (5.3) form a closed system of equations to determine the four unknown functions u_α, u, ψ (where $u_\alpha=r_\alpha{}^iu_i$, $u=n^iu_i$). The boundary conditions on Γ for (5.2), (5.3) and (5.4) have the form

$$\begin{gathered}(T^{\alpha\beta}-b_\lambda{}^\alpha M^{\lambda\beta})\nu_\beta=0\\ M^{\alpha\beta}_{;\alpha}\nu_\beta+\frac{d}{ds}(M^{\alpha\beta}\nu_\alpha\tau_\beta)=0,\quad M^{\alpha\beta}\nu_\alpha\nu_\beta=0\\ \psi=\varphi_n \ \text{on}\ \Gamma_\varphi^{(n)},\quad n=1,\ldots,N,\quad G^\alpha\nu_\alpha=0\ \text{on}\ \Gamma_\tau\end{gathered} \tag{5.5}$$

where τ_α, ν_α are components of the surface vectors, tangent and normal to the contour Γ.

Case B. The equilibrium equations and boundary conditions are the same as in the classical theory of elastic shells /2/. Changes concern just the equations of state

$$\begin{gathered}S^{\alpha\beta}=\frac{\partial\Psi}{\partial A_{\alpha\beta}}=h\left(c_P^{\alpha\beta\gamma\delta}A_{\gamma\delta}+e_P^{3\alpha\beta}\frac{\varphi_0}{h}\right)\\ M^{\alpha\beta}=-\frac{\partial\Psi}{\partial B_{\alpha\beta}}=-\frac{h^3}{12}c_N^{\alpha\beta\gamma\delta}B_{\gamma\delta}\end{gathered} \tag{5.6}$$

6. Connection between the three- and two-dimensional electroelastic states of stress. To complete the construction of the piezoelectric shell model, we indicate a method of restoring the three-dimensional electroelastic state of stress by means of the two-dimensional state. To do this, the strain $\boldsymbol{\varepsilon}$ and the electric field $\mathbf{E}$ must be found to a first approximation by the asymptotic formulas in Sect.4. The stress tensor $\boldsymbol{\sigma}$ and the induction vector $\mathbf{D}$ are found by means of the three-dimensional electroelasticity relationships (1.9). Omitting the calculations, we present the final formulas for $(\boldsymbol{\varepsilon},\mathbf{E})$ and $(\boldsymbol{\sigma},\mathbf{D})$ to a first approximation.

Case A. Strain-electrical field

$$\begin{gathered}\varepsilon_{\alpha\beta}=A_{\alpha\beta}-B_{\alpha\beta}\xi\\ 2\varepsilon_{\alpha3}=-(r_\alpha^{\mu\nu}A_{\mu\nu}-t_\alpha{}^\mu F_\mu)+t_\alpha^{\mu\nu}B_{\mu\nu}\xi\\ \varepsilon_{33}=-(r^{\alpha\beta}A_{\alpha\beta}-r^\alpha F_\alpha)+r^{\alpha\beta}B_{\alpha\beta}\xi\\ E_\alpha=F_\alpha,\ E_3=-(q^{\alpha\beta}A_{\alpha\beta}+q^\alpha F_\alpha)+q^{\alpha\beta}B_{\alpha\beta}\xi\end{gathered} \tag{6.1}$$

Stress-induction

$$\begin{gathered}\sigma^{\alpha\beta}=c_N^{\alpha\beta\gamma\delta}A_{\gamma\delta}-e_N^{\gamma\alpha\beta}F_\gamma-c_N^{\alpha\beta\gamma\delta}B_{\gamma\delta}\xi=\frac{S^{\alpha\beta}}{h}+\frac{12}{h^3}M^{\alpha\beta}\xi\\ \sigma^{\alpha3}=0,\quad \sigma^{33}=0\\ D^\alpha=e_N^{\alpha\beta\gamma}A_{\beta\gamma}+\epsilon_N^{\alpha\beta}F_\beta-e_N^{\alpha\beta\gamma}B_{\beta\gamma}\xi=\frac{G^\alpha}{h}-e_N^{\alpha\beta\gamma}B_{\beta\gamma}\xi,\quad D^3=0\end{gathered} \tag{6.2}$$

Case B. Strain-electrical field

$$\varepsilon_{\alpha\beta}=A_{\alpha\beta}-B_{\alpha\beta}\xi \tag{6.3}$$
$$2\varepsilon_{\alpha 3}=-k_\alpha\frac{\varphi_0}{h}-k_\alpha^{\mu\nu}A_{\mu\nu}+t_\alpha^{\mu\nu}B_{\mu\nu}\xi$$
$$\varepsilon_{33}=-f\frac{\varphi_0}{h}-f^{\alpha\beta}A_{\alpha\beta}+r^{\alpha\beta}B_{\alpha\beta}\xi$$
$$E_\alpha=0,\quad E_3=-\frac{\varphi_0}{h}+q^{\alpha\beta}B_{\alpha\beta}\xi$$

Stress-induction

$$\sigma^{\alpha\beta}=c_P^{\alpha\beta\gamma\delta}A_{\gamma\delta}+e_P^{3\alpha\beta}\frac{\varphi_0}{h}-c_N^{\alpha\beta\gamma\delta}B_{\gamma\delta}\xi=\frac{S^{\alpha\beta}}{h}+\frac{12}{h^3}M^{\alpha\beta}\xi \tag{6.4}$$
$$\sigma^{\alpha 3}=0,\quad \sigma^{33}=0$$
$$D^\alpha=e_P^{\alpha\beta\gamma}A_{\beta\gamma}-\epsilon_P^{\alpha 3}\frac{\varphi_0}{h}-e_N^{\alpha\beta\gamma}B_{\beta\gamma}\xi,\quad D^3=-\epsilon_P^{33}\frac{\varphi_0}{h}+e_P^{3\alpha\beta}A_{\alpha\beta}$$

7. Error estimates of the classical theory of piezoelectric shells. We consider the linear vector space of electroelastic states of stress that consists of elements of the form $\Xi=(\sigma,\mathbf{E})$, where σ is the stress field, and $\mathbf{E}$ is the electric field in the domain V. In this space we introduce the following norm:

$$\|\Xi\|_{L_2}^2=C_2[\Xi]=\int_V G(\sigma,\mathbf{E})\,dv \tag{7.1}$$

where the function $G(\sigma,\mathbf{E})$ is the Legendre transformation of the function $U(\varepsilon,\mathbf{D})$ in all the variables /3/

$$G(\sigma,\mathbf{E})=\frac{1}{2}s_{abcd}^E\sigma^{ab}\sigma^{cd}+d_{abc}\sigma^{bc}E^a+\frac{1}{2}\epsilon_{ab}^T E^a E^b \tag{7.2}$$

We will give the description "kinematically allowable" to those electroelastic states of stress Ξ° for which the strain ε° and the electric induction $\mathbf{D}^\circ$ fields exist such that

$$\overset{\circ}{\varepsilon}_{ab}=x^i_{(a}\overset{\circ}{w}_{i,b)};\quad \nabla_a D^{\circ a}=0,\quad D^{\circ a}\nu_a=0\quad\text{on}\quad S_\tau \tag{7.3}$$

while σ° and $\mathbf{E}^\circ$ are expressed in terms of ε° and $\mathbf{D}^\circ$ according to (1.4). We call those $\Xi^\wedge$ "statically allowable" for which

$$\nabla_b\sigma^{\wedge ab}=0,\quad \sigma^{\wedge ab}\nu_\beta=0\quad\text{on}\quad\partial V \tag{7.4}$$
$$E_a^\wedge=-\nabla_a\varphi^\wedge,\quad \varphi^\wedge=\varphi_n\quad\text{on}\quad S_\varphi^{(n)},\quad n=1,\ldots,N.$$

Let Ξ be a real electroelastic state of stress that is realized in a piezoelectric body V on specifying values of the potential φ_n on the electrodes $S_\varphi^{(n)}$. Thus the following identity

$$C_2[\Xi-{}^1\!/_2(\Xi^\circ+\Xi^\wedge)]=C_2[{}^1\!/_2(\Xi^\circ-\Xi^\wedge)] \tag{7.5}$$

that generalizes the Prager-Synge identity /7/ to the statics of piezoelectrics and is provable by an analogous method, turns out to be valid.

From the identity (7.5) we have

Theorem. The electroelastic state of stress constructed by the two-dimensional theory of piezoelectric shells (Sects.5 and 6) differs in the norm L_2 from the exact electroelastic state of stress constructed by the three-dimensional theory of piezoelectricity by a quantity of the order of h_*+h_{**} as compared with one.

To prove this we must present the kinematically and statically allowable three-dimensional fields of the electroelastic states of stress that differ from that constructed by the two-dimensional theory by a quantity of the order of h_* and h_{**} as compared with unity /6, 8/.

Case A. Kinematically allowable field. We determine the displacement by means of (4.1) and (4.7), and ε° by means of (7.3). We take $D^{\circ\alpha}$ in the form $D^{\circ\alpha}=z^\alpha(\xi^\beta)-e_N^{\alpha\beta\gamma}B_{\beta\gamma}\xi$ for the induction vector $\mathbf{D}^\circ$. (Later, all quantities without the $^\circ$ and $^\wedge$ refer to solutions of the equilibrium equations of piezoelectric shells by the two-dimensional theory of Sect.5.) We select the quantity z^α such that $\langle D^{\circ\alpha}\varkappa\rangle_\xi=e_N^{\alpha\beta\gamma}A_{\beta\gamma}+\epsilon_N^{\alpha\beta}F_\beta\equiv G^\alpha/h$. The component $D^{\circ 3}$ is found from the known $D^{\circ\alpha}$ from the solution of the electrostatics equation

$$(D^{\circ\alpha}\varkappa)_{;\alpha}+(D^{\circ 3}\varkappa)_{,\xi}=0 \tag{7.6}$$

Knowing $(\varepsilon^\circ,\mathbf{D}^\circ)$, we find Ξ° from the electroelasticity Eqs.(1.4).

Statically allowable field. We will write down the mechanical equilibrium equations for a shell /2/

$$\tau^{\alpha\beta}_{;\beta}+\frac{\partial}{\partial\xi}(\mu_\beta{}^\alpha\tau^\beta)-\tau^\beta b_\beta{}^\alpha=0,\quad \tau^\beta_{;\beta}+\tau^{\alpha\beta}b_{\alpha\beta}+\frac{\partial}{\partial\xi}\tau=0 \tag{7.7}$$
$$(\tau^{\alpha\beta}=\mu_\lambda{}^\alpha\sigma^{\lambda\beta}\varkappa,\ \tau^\alpha=\sigma^{\alpha 3}\varkappa,\ \tau=\sigma^{33}\varkappa,\ \mu_\beta{}^\alpha=\delta_\beta{}^\alpha-b_\beta{}^\alpha\xi)$$

To find the stress tensor $\sigma^{\wedge}$ satisfying (7.7), we proceed thus. We give $\sigma^{\wedge\alpha\beta}$ in the form $\sigma^{\wedge\alpha\beta} = s_0^{\alpha\beta} + \xi s_1^{\alpha\beta}$, where $s_0^{\alpha\beta}, s_1^{\alpha\beta}$ are independent of ξ and are selected from the condition $\langle\tau^{\wedge\alpha\beta}\rangle_\xi = T^{\alpha\beta}, \langle\tau^{\wedge\alpha\beta}\xi\rangle_\xi = M^{\alpha\beta}$. Solving (7.7), we can find $\tau^{\wedge\alpha}$ and $\tau^{\wedge}$ and then $\sigma^{\wedge\alpha 3}$ and $\sigma^{\wedge 33}$. It turns out that (5.2) is the sufficient conditions for the existence of $\tau^{\wedge\alpha}$ and $\tau^{\wedge}$. We specify the potential $\varphi^{\wedge}$ by formulas (4.1), (4.7). It is assumed here that the three-dimensional boundary conditions are given not in the form (2.2), but in conformity with (4.1) and (4.7) (the so-called regular boundary conditions in the terminology of /8/).

Considerations concerning the generalization of the Saint-Venant principle to piezoelectrics /18/.

Case B. *Kinematically allowable field.* The displacements $w^{\circ i}$ are given by (4.9) and (4.13), $D^{\circ\alpha}$ by (6.4), and $D^{\circ 3}$ is found from (7.6). We hence find $\mathbf{S}^{\circ}$ by means of (1.4).

Statically allowable field. The construction is analogous to case A.

It can be shown that in both cases $\mathbf{S}^{\circ}$ and $\mathbf{S}^{\wedge}$ differ from that constructed by the two-dimensional theory (Sect.5) by a quantity of the order of $h_* + h_{**}$ as compared with unity. From the above and the identity (7.5) there follows the asymptotic accuracy of the constructed theory in the energy norm (7.1).

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